Symmetric Union Presentations for 2-Bridge Ribbon Knots

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Abstract

Symmetric unions have been defined as generalizations of Kinoshita-Terasaka's construction in 1957. They are given by diagrams which look like the connected sum of a knot and its mirror image with additional twist tangles inserted near the symmetry axis. Because all symmetric unions are ribbon knots, we can ask how big a subfamily of ribbon knots they form. It is known that all 21 ribbon knots with crossing number less or equal 10 are symmetric unions.

In this talk we extend our knowledge about symmetric unions: we prove that the family of symmetric unions contains all known 2-bridge ribbon knots. The question, however, whether the three families of 2-bridge ribbon knots, found by Casson and Gordon in 1974, are a complete list of all 2-bridge ribbon knots, is still open.

1 Ribbon knots: definition

Definition 1.1
A knot K in S^3 is a *ribbon knot* if it bounds an immersed disk in S^3 with only ribbon singularities.

Recall that K is a *slice knot* if it bounds a smoothly embedded 2-disk D^2 in B^4. Every ribbon knot is slice. A notorious question in knot theory is whether the converse is true.

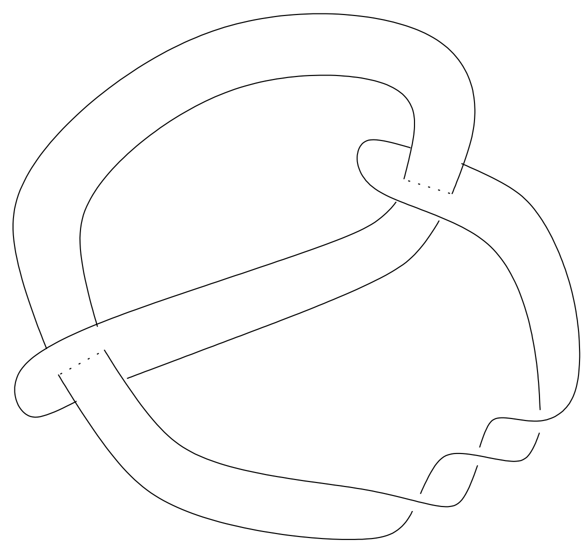

Figure 1: Example of ribbon singularities

2 Presentations of ribbon knots

We know all ribbon knots with crossing number ≤ 10: there are 21 of them.

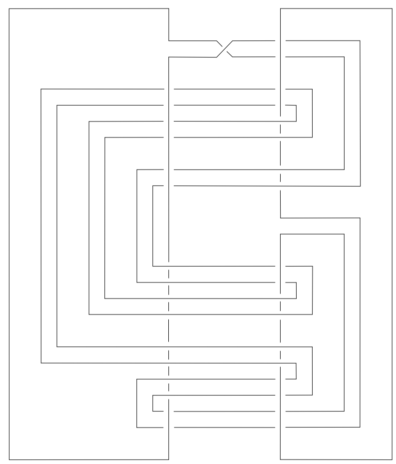

Figure 2: A ribbon presentation for the knot 10_{87} taken from "A survey of knot theory"

Symmetric unions yield another way of showing that knots are ribbon.

Theorem 2.1
All 21 ribbon knots with crossing number ≤ 10 are symmetric unions. □

As an example we show the symmetric union for 10_{87} which was missing from my list in 1998.

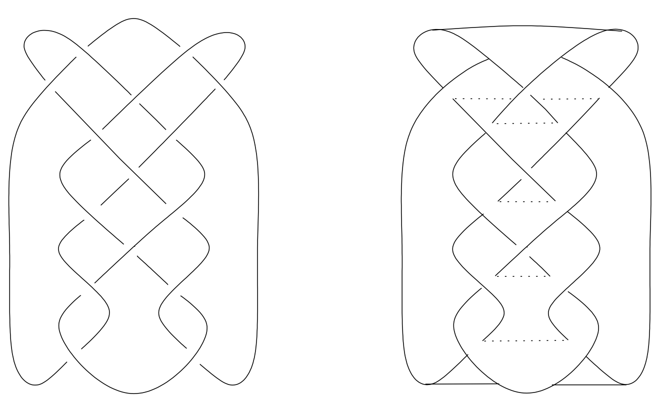

Figure 3: The knot 10_{87} as a knot diagram and as a symmetric ribbon with twists

3 Symmetric unions: definition

Definition 3.1
Let D be an unoriented knot diagram and D^* the diagram D reflected at an axis in the plane. If, as in Fig. 4, we insert the tangle $\asymp$ and twist tangles n_i we call the result a *symmetric union* of D and D^* with twist parameters n_i, $i = 1, \ldots, k$. The *partial knot* $\hat{K}$ of the symmetric union is the knot given by the diagram D.

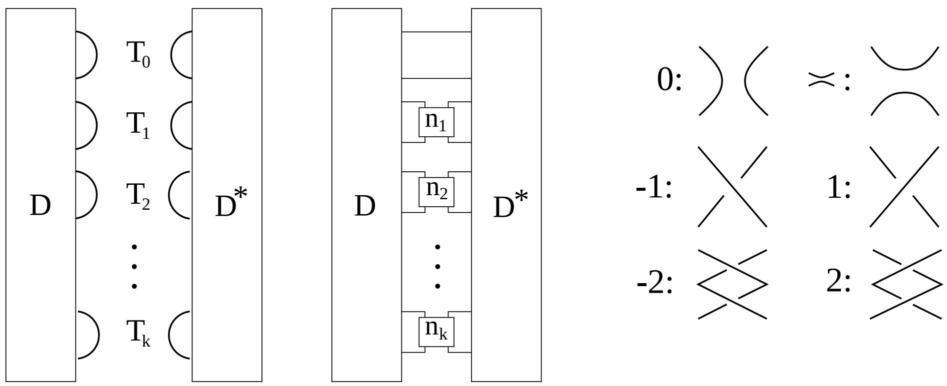

Figure 4: Definition of symmetric unions

Examples:
1.) As an example we consider again the diagram in Fig. 5. Here the partial knot is the knot 6_1:

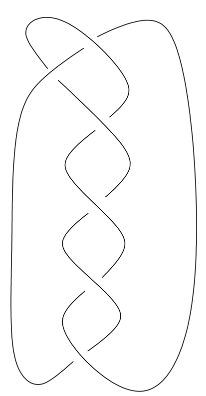

Figure 5: Partial knot of the symmetric union diagram for 10_{87} in Fig. 5

2.) If all $n_i = 0$ then we get the well-known symmetric ribbon for $\hat{K}\sharp - \hat{K}^*$.

4 Properties of symmetric unions

The properties in Theorem 4.1 were already known to Kinoshita and Terasaka in the case that $k = 1$ (only one twist tangle inserted). The theorem is proved for instance by using the matrix definition of the Alexander polynomial and the Goeritz matrix.

Theorem 4.1
The Alexander polynomial of a symmetric union depends only on the parity of the twist numbers n_i.
The determinant of a symmetric union K is independent of the twist numbers and equals the square of the determinant of the partial knot: $\det(K) = \det(\hat{K})^2$. □

Analyzing the knot groups (modulo squares of meridians) of K and $\hat{K}$ we can prove:

Theorem 4.2
If the partial knot of a symmetric union K is non-trivial, then K is non-trivial. □

Partial knots are not unique: different partial knots can result in the same symmetric union.

In this talk we are interested mainly in the ribbon property of symmetric unions:

Theorem 4.3
Symmetric unions are ribbon knots. □

The proof uses the same construction as for $K\sharp - K^*$ with additional twists in the ribbon, see Fig. 6.

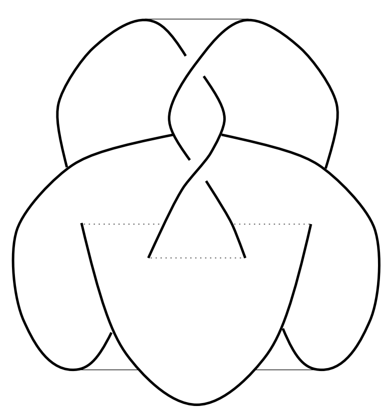

Figure 6: Symmetric unions are ribbon knots

5 Use of symmetric unions

- **Knots with $\Delta = 1$ or** $\det = 1$**:** The knot 10_{153} and the Kinoshita-Terasaka knot are symmetric unions of the trivial knot, see Fig. 7. Hence they have determinant 1. For the Kinoshita-Terasaka knot the twist parameter is an even number, hence its Alexander polynomial is equal to 1.

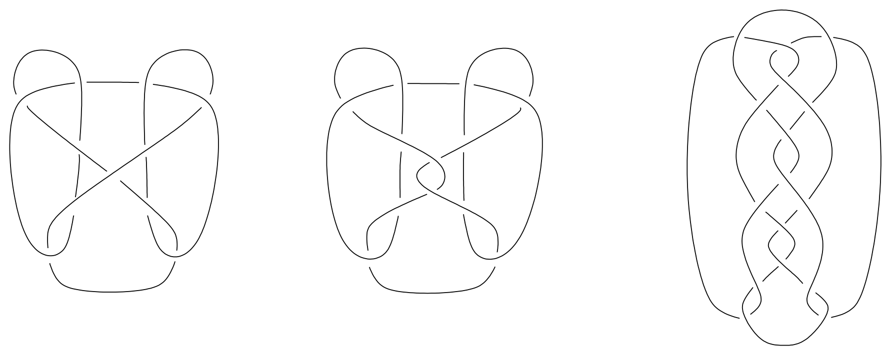

Figure 7: The knot 10_{153}, the Kinoshita-Terasaka knot and the example of Tanaka

- **Families of knots with the same polynomial invariants:** T. Kanenobu used in 1986 symmetric unions to construct knots with the same Jones or Homfly polynomial. In 2004 he used the following chiral knot in order to present a knot whose chirality is not detected by the Links/Gould invariant.

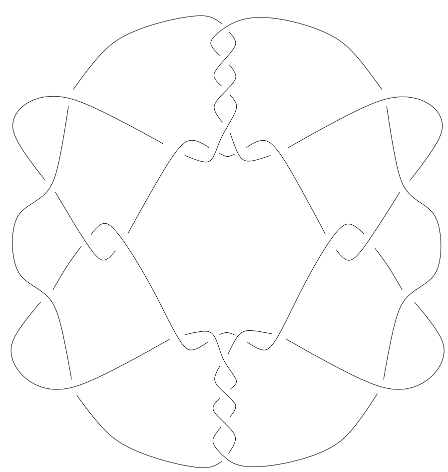

Figure 8: A symmetric union with two symmetry axis used by Kanenobu

- **Tanaka's example:** Toshifumi Tanaka used in 2004 a symmetric union to construct a counterexample for a conjecture of T. Fiedler on the Jones polynomial of ribbon knots, see Fig. 7

6 2-Bridge knots: definition

Recall the definition of the bridge number of a knot: if $v \in \mathbb{S}^2$ is a unit vector in $\mathbb{R}^3$ and K is a knot, then let $b_v(K)$ be the number of maxima of the orthogonal projection of K on the line spanned by v. Then the *bridge number* of K is:

$$b(K) := \min_{K' \sim K} \min_{v} b_v(K)$$

Because knots with bridge number 1 are trivial, the simplest cases occur for bridge number 2. They were studied by

- Bankwitz/Schumann (Viergeflechte, 1934),
- Schubert (Knoten mit 2 Brücken, 1956) and
- Conway (rational knots, 1970).

Their double branched coverings are lens spaces $L(p,q)$ with p equal to the determinant of the knot.

In the plait normal form $C(a_1,\ldots,a_n)$, also called *Conway notation*, the numbers of half-twists a_i are related to the parameters p and q by the continued fraction $[a_1,\ldots,a_n] = \frac{p}{q}$.

We sometimes call $\frac{p}{q}$ the "fraction" of K.

7 The result of Casson/Gordon on 2-bridge ribbon knots

The following theorem gives a necessary condition for 2-bridge ribbon knots.

Theorem 7.1 (Casson/Gordon, 1974)
Let K be a ribbon knot with $b(K) = 2$ and double branched covering $L(p^2, q)$. Then:
$\sigma(p, q, r) := 4(\text{area}\Delta(pr, \frac{qr}{p}) - \text{int}\Delta(pr, \frac{qr}{p})) = \pm 1, \quad \forall r = 1, \ldots, p-1.$ □

Here $\Delta(pr, \frac{qr}{p})$ denotes the triangle $((0,0),(pr,0),(pr,\frac{qr}{p}))$.

The value of $\text{int}\Delta(pr, \frac{qr}{p}))$ is computed by counting lattice points similar to Pick's formula: interior points count as 1, boundary points as 1/2 and vertices different from (0,0) as 1/4.

In the example of Figure 9
- for $r = 1$ the area of the triangle is 23 and the value of $\text{int}\Delta(pr, \frac{qr}{p}))$ is $23\frac{1}{4}$
- for $r = 2$ these values are 92 and $91\frac{3}{4}$.

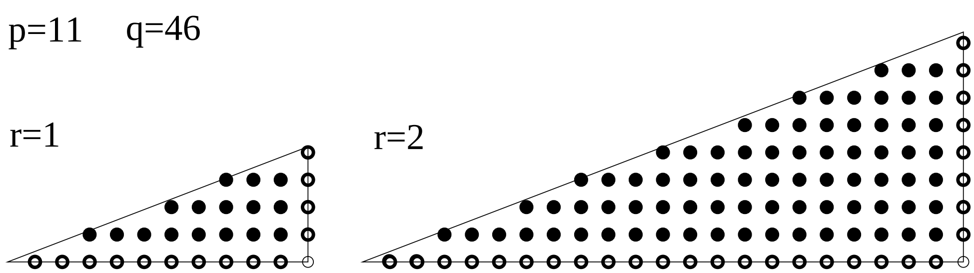

Figure 9: Computation of $\text{int}\Delta(pr, \frac{qr}{p}))$

8 The 3 families of Casson/Gordon satisfying the necessary condition

All knots found to satisfy the necessary condition are the following 2-bridge knots (and their mirror images):

- Family 0: $C(a, b, \ldots, w, x, x+2, w, \ldots, b, a)$, with parameters > 0,
- Family 1: $C(2a, 2, 2b, -2, -2a, 2b)$, with $a, b \neq 0$,
- Family 2: $C(2a, 2, 2b, 2a, 2, 2b)$, with $a, b \neq 0$,

and each knot in this list is a ribbon knot. Hence this is a complete list of known 2-bridge ribbon knots. It is a conjecture (since 1974) that there are no other 2-bridge ribbon knots.

Remark 8.1
1.) Amphicheiral 2-bridge knots have (an even length) palindromic Conway notation: for instance $C(2, 1, 1, 2)$. Hence for each $c \geq 4$ there are exactly as many knots of family 0 with minimal crossing number $c+2$ as there are amphicheiral 2-bridge knots with minimal crossing number c.
2.) Families 1 and 2 are not disjoint: knots of family 1 with $a = -1$ equal those of family 2 with $a = -1$ or $b = -1$.

9 Symmetric union presentations for the 3 families

Our main Theorem is:

Theorem 9.1
Every knot contained in one of the three families is a symmetric union.

Proof: The proof is given by the following deformations of knot diagrams. For family 0 we have

$$C(a,b,\ldots,w,x+1,x-1,w,\ldots,b,a) = C(a,b,\ldots,w,x,1,-x,-w,\ldots,-b,-a)$$

which is a symmetric union (this family was already considered by Kanenobu in 1986).

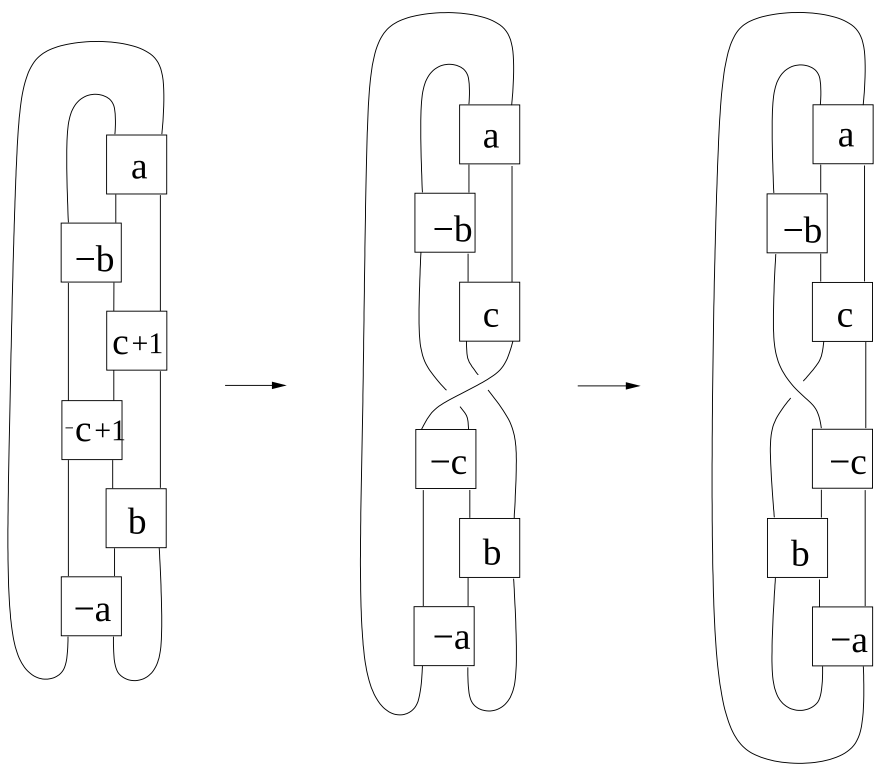

Figure 10: Proof for family 0

For families 1 and 2 we have the following diagrams:

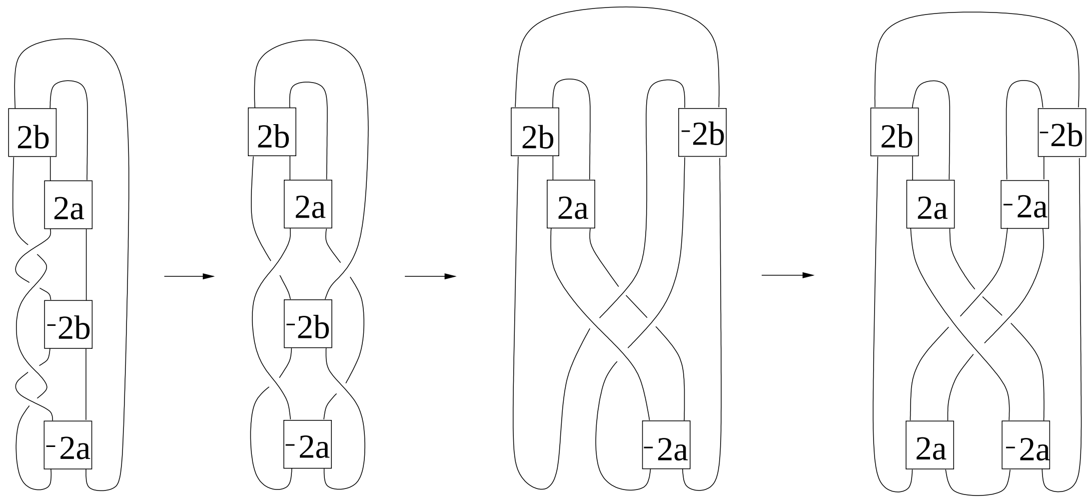

Figure 11: Proof for family 1

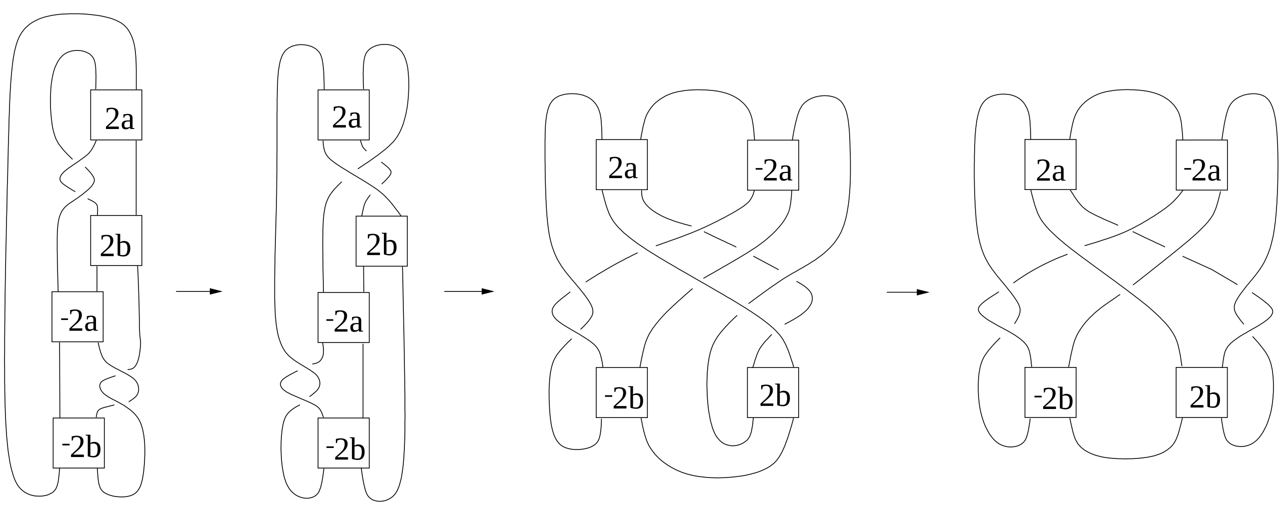

Figure 12: Proof for family 2

□

The surprise is that it is necessary to insert extra 2a/-2a twists to obtain the symmetric diagram for family 1.

10 Relationship between ribbon knot and its partial knot

There is another characterisation of the three families (this is exercise 18 e) in the unpublished notes of L. Siebenmann: *Exercices sur les noeuds rationnels*):

Lemma 10.1 (1975)
The following conditions for a 2-bridge knot K with double branched cover $L(p^2, q)$ are equivalent:

a) K is contained in one of the three families of 2-bridge ribbon knots of Casson and Gordon.

b) The parameter q satisfies

i) $q = np \pm 1$, with $(n, p) = 1$, or

ii) $q = n(p \pm 1)$, with $n|2p \mp 1$, or

iii) $q = n(p \pm 1)$, with $n|p \pm 1$, n odd, or

iv) $q = n(2p \pm 1)$, with $d \cdot n = p \mp 1$, d odd.

$\square$

Example: Several of the conditions in b) can apply. For the knot $\frac{121}{84}$ we have simultaneously $84 = 7(11 + 1)$ as in ii) and $84 = 4(2 \cdot 11 - 1)$ as in iv).

This description allows an elegant computation of the partial knot's fraction in the symmetric union representation in Theorem 9.1.

Theorem 10.2
Let K be a knot contained in one of the three families and $\hat{K}$ be its partial knot as constructed in Theorem 9.1. If $K = \frac{p^2}{q}$ then $\hat{K} = \frac{p}{n}$ where n is as defined in Lemma 10.1. Different possibilities for n result in the same partial knot. $\square$

Example: By ii) for $\frac{121}{84}$ we get the partial knot $\frac{11}{7}$. By iv) we get the partial knot $\frac{11}{4}$. These fractions denote the same knot $C(2, 1, 3)$.

Questions: Is there a geometric explanation for the result in Theorem 10.2? Are partial knots unique for 2-bridge ribbon knots?

11 Open questions and further projects

Conjecture 11.1 (1974)
Every 2-bridge ribbon knot is contained in one of the three families of Casson and Gordon.

We found that the conjecture is true for all 2-bridge knots with det $\leq 571^2$ (which includes all 2-bridge knots with crossing number ≤ 26) by computing the necessary condition.

For the number of 2-bridge ribbon knots for each crossing number $3 \leq c \leq 19$ we found (knots which belong to both family 1 and family 2 are shown together with family 1; the list can be extended to $c \leq 26$):

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
family 0				1		1		3		5		11		21		43	
family 1						1	1	1	1	2	2	2	2	3	3	3	3
family 2										1	1			2	2		
total				1		2	1	4	1	8	3	13	2	26	5	46	3

Projects:

1. Use Eisenstein method mentioned in the article of Siebenmann to improve computation.

2. All ribbon knots and symmetric unions with minimal crossing number ≤ 10 are known. Extend this list to knots with minimal crossing number 11 (is Conway knot ribbon?).

3. Prove Conjecture 11.1.

4. Prove Conjecture 11.2.

Conjecture 11.2
Every ribbon knot is a symmetric union.

or find a ribbon knot which is not a symmetric union!

References

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